

THESIS FOR THE DEGREE OF LICENTITATE

Game theory and applications

—connecting individual-level interactions to
collective-level consequences

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Abstract

Individual-level interactions and decisions spread through populations and change the collective-level dynamics in an intricate way. Nevertheless, game theory is well suited for the unification of these viewpoints. This thesis introduces the works *The strength of diversity* and *Decisions and disease: a mechanism for the evolution of cooperation*, which show that games have broad applications in population dynamics modeling.

In *The strength of diversity* we calculate the equilibrium strategies of the so-called game of teams. The game of teams is an individual-level competition between teams, and a team's strategy in this context is a distribution of strength over the team members. It turns out that the equilibrium strategies are flat distributions or 'alternating' flat distributions whenever there exists equilibrium strategies.

In *Decisions and disease: a mechanism for the evolution of cooperation* we combine the classic SIR and SIS models from epidemiology with the prisoner's dilemma game. The transmission rate is computed as the average over defecting and cooperating individuals, and the individuals are subjected to a replicator equation that takes into account the portion of infectious members of the population. We compute the steady state solutions and interpret the results.

Keywords: game theory, non-cooperative game theory, prisoner's dilemma, equilibrium strategy, disease, distribution, compartmental model, epidemiological model,

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List of publications

This thesis consists of an extended summary and the following appended manuscripts.

Paper 1: Rowlett, J., Karlsson, C. & Nursultanov, M. (2021). The strength of diversity. <https://arxiv.org/abs/2108.04054v2>

Paper 2: Karlsson, C. & Rowlett, J. (2020). Decisions and disease: a mechanism for the evolution of cooperation. *Nature Scientific Reports*, 10:13113, <https://doi.org/10.1038/s41598-020-69546-2>

Author contribution:

Paper 1: I did many of the pen-and-paper calculations, especially on the discrete game, and I contributed to the literature study and on writing the manuscript.

Paper 2: I did a literature study, finding the connection between the SIR model and the dynamic parameter β , contributed to writing the manuscript, and I did the numerical study as well as many of the pen-and-paper calculations.

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Notation, abbreviations and nomenclature

\mathbb{Z}	The integers ..., -2 , -1 , 0 , 1 , 2 , ...
\mathbb{R}	The real numbers
a.e.	Almost Everywhere
game	A collection of players, payoff functions and strategies
equilibrium	A state where no player has incentive to change strategy
E	Expectation, see §2.2
p	Expectation of normalized strategies, payoff
CA	Competitive Ability
MCA	Mean Competitive Ability
C	The upper bound on MCAs; the cost in Donor-Recipient games
SIR	A compartment model with categories susceptible, infectious and removed
SIS	A compartment model with categories susceptible and infectious. Typically, this models a disease without immunity.

Chapter 1

Games, members and populations

Game theory is the appropriate tool whenever the success of an individual depends on others.

—Martin A. Nowak and Karl Sigmund (*Science*, 2004)

Game theory is the study of decision-making under conflicting interests. When the actions undertaken by a decision-maker affects the considerations of other ‘players’ there is a mathematical procedure to find the overall ‘best choices’ and also the rational choices from each player’s perspective. Oftentimes, they differ; the overall best is never achieved because it does not seem rational from an individual perspective. These insights are highlighted by game theory thanks to its simple rules. Although simple, game theory analyses are an essential part of modern decision-making. Today, agreements such as the Kyoto protocol (on reducing the climate footprint worldwide) undergo game theory analysis. Economy, negotiation, trade, resource management, ecology and physics are just some examples of applications of the modern game theory that was formalized during the 20th century. (Mazalov, 2014)

Although games were discussed systematically long before, certainly already in ancient times (Ross, 2021), the starting point of today’s rigorous

treatment of game theory and applications is widely considered the book *The Theory of Games and Economic Behaviour* (1944) by J. von Neumann and O. Morgenstern. Later pioneers such as J. Nash, L. Shapley, J. Maynard Smith and others have refined the theory and expanded its applicability.

This thesis comprises the two articles *The strength of diversity* and *Decisions and disease: a mechanism for the evolution of cooperation*, which are both fundamentally oriented investigations that show how individual-level interaction and collective-level dynamics are unified by means of game theory. These papers do also show the generality of applications. The results are abstract and not restricted to a certain context or interpretation.

1.1 The early days of game theory

We begin this chapter with one of the most important results of game theory, namely an explanation of *the tragedy of the commons*. In a group of decision-makers, the tragedy of the commons is in general a situation where no participant seeks to cooperate for the common good even though such cooperation would benefit every participant.

1.1.1 A motivating example in game theory

Climate change is a major issue that has led to much negotiation between countries. Consider a meeting between two countries that can either defect from an issued agreement or decide to cooperate within the agreement. If both cooperate, they each get a climate benefit of 6 (in some unit, e.g. a monetary currency or natural resources), whereas if only one of them cooperates they receive a lower climate benefit of 3. Cooperating costs 4. This leads to the following total payoffs: If both abate, both get $6 - 4 = 2$. If both defect, both get $0 - 0 = 0$. If only one abates, the abater gets $3 - 4 = -1$, and the defector gets $3 - 0 = 3$. Now each country has the following options to consider: If the other country decides to cooperate, then defecting guarantees a benefit of 3 while cooperating gives 2. If the other country decides to defect, then defecting comes with a benefit of 0 and cooperating gives -1 . That is, if the other's strategy is not known

it is always better to defect. The same goes for the other country. The rational choice is therefore to defect, not abate. This situation – both players defect as they do not trust the other player to cooperate even though the common good would benefit most if both would cooperate – is the essential mechanism of the tragedy of the commons.

It is standard procedure to write these possible outcomes in a matrix as in the figure below. Here, *C* denotes the option ‘cooperate’ and *D* denotes ‘defect.’ Each box contains the gain to each player such that the top-right number belongs to the player *B*. The above example is symmetric since both players have the same set of actions and are paid the same payoffs.

		player <i>B</i>	
		<i>C</i>	<i>D</i>
player <i>A</i>	<i>C</i>	2 2 2 -1	3 3 -1 0
	<i>D</i>	3 -1 0 0	0 3 0 0

The concept of rational decision from an individual’s point of view is captured mathematically by the notion of *equilibrium*. In an equilibrium, no player has incentive to change strategy. Borel formulated and studied game equilibria in the early 20th century for two-player games and same did von Neumann during the 20’s though the 40’s. The equilibrium concept was extended to include n players by J. Nash in the 50’s. (Hammond, 2003) Nash proved that any game with n players has a symmetric equilibrium (Nash, 1951) and a game’s equilibrium is therefore often called *Nash equilibrium*. In the above example, there is an equilibrium when both players defect.

A. Tucker named the above game the *prisoners’ dilemma* as a way to popularize Nash’s ideas to the psychology community. M. Flood and M. Dresher discussed games with the structure of prisoner’s dilemma in 1950, but neither they nor Nash published their ideas immediately. (Kuhn, 2019)

In this thesis we consider *repeated games* with the possibility to change action when a new turn is played as long as there is a fixed probability distribution of choices. Such a distribution was named a *mixed strategy* by Nash. In the example above there were two options (C and D) for each player, so if a player plays C with probability x then the same player chooses D with probability $1 - x$. In this case we can compute what payoff is expected, statistically. The equilibrium of a game is then defined as above with the assumption that each player wants to maximize its expected payoff. A *pure strategy* assigns probability 1 to only one option.

In *non-cooperative* game theory, players do not reveal their intentions to other players and they therefore act to maximize the individual payoff that they receive. In contrast to this, cooperative games follow a different set of rules. There are no individual payoffs but the game has an overall value which is determined by the values of the ‘coalitions’ that form in the game. This thesis will only concern non-cooperative games, but the interested reader is encouraged to consult the founding work by Shapley (1953) for an introduction to cooperative games.

Chapter 2 of this thesis investigates the Nash equilibria of a game of competing teams. In Chapter 3, the above symmetric matrix plays a particularly important role in the context of changing behaviors during a disease outbreak.

1.1.2 A motivating example in population dynamics

‘How can one explain such oddities as snakes that wrestle with each other, deer that refuse to strike “foul blows,” and antelope that kneel down to fight?’ asked biologists Maynard Smith and Price (1973). They used game theory and their newly developed notion of *evolutionary stable strategies* to explain why fighting between members of the same species usually do not escalate unless the opponent aim to cause severe injury. In this context, the strategies ‘cooperate’ and ‘defect’ are termed ‘conventional conflict’ and ‘dangerous conflict’ making the matrix-game formulation seen in the prisoner’s dilemma suitable.

In population dynamics games there is an evolution of strategies which are designs of a selection process during repeated games rather than the ‘rational choices’ in Nash equilibria. This selection process can result in strategies that are different from the classical game from the previous section. Maynard Smith and Price (1973) performed computer simulations of repeated matrix-games and realized that the selection process leads to a classic equilibrium if in addition to Nash’s equilibrium condition there is a stability condition (Riechert and Hammerstein, 1983). A year later, Maynard Smith (1974) designed a mechanism that in some games displays the development from non-optimal strategies to the stable ones by means of an evolutionary process. We will encounter a version of these ideas in Chapter 3 when we consider games in combination with epidemiological models and identify the stable steady state solutions.

Chapter 2

The Game of Teams

Talent wins games, but teamwork and intelligence win championships.

—Michael Jordan (in *I Can't Accept Not Trying*. 1994, Harper)

The Game of Teams has been developed during the last couple of years as a model to explain why certain compositions of team members are more advantageous than others. Central to this game is the assumption that competition occurs on an individual level whereas strategies are defined on a team-level. This applies in a wide range of situations such as economy (portfolio theory, management etc.), research, sports, behaviorological sciences and ecology. It has been demonstrated that diversity is beneficial in all these contexts. Results from ecology shows that diversity is healthy for a biological system, for instance as protection against extinction of species. Similarly, a diverse portfolio of investments protects the investor from financial backlash in case individual investments are unstable. Teams of scientists perform better if they are diverse in all senses they can be.

The game of teams do not treat teams as individuals and herein lies its importance. We construct a simple game in this chapter and we show that the success of a team depends heavily on the distribution of resources among its members. This provides an explanation to the strength of diversity without restricting the results to a specific context

(e.g. ecology.)

2.1 Foundations of non-cooperative games

The purpose of game theory was originally to explain conscious and rational decision making mathematically. Therefore it is assumed that all players in a game act to maximize their payoffs, with no other intentions than that. In a *zero-sum game* the gain to one player exactly equals the loss of the rest of the players. If all gains are added up (counting loss as negative gain), the sum is zero.

Strategies and equilibrium.

There are two formulations for two-person zero-sum games with a finite number of alternatives for each player: the normal form and the extensive tree form (Hammond, 2003). The normal form is also called the matrix form and we encountered this in §1.1.1 in the discussion on climate agreements. Only the normal form games are considered in this thesis. These games are determined by a static set of options and a payoff function for each player. A payoff is a mapping to the real numbers that depends on the other players' decisions.

A *strategy* is a rule for decision-making such that the payoffs in the game are determined once the strategies of all players are known. Let G denote the set of strategies. It may possess any structure (a finite set of values, a subset of \mathbb{R}^n , a set of measurable functions, etc.)

An equilibrium is defined as a set of strategies such that no player has incentive to change strategy, assuming the other players' strategies are unchanged. If we let $\{f_i\}_{i=1}^n$ denote a set of strategies and p_i denotes the payoff to player i , then $\{f_i\}_{i=1}^n$ is an equilibrium if for every i

$$p_i(f_1, \dots, f_n) = \max_{g_i \in G} p_i(f_1, \dots, f_{i-1}, g_i, f_{i+1}, \dots, f_n). \quad (2.1)$$

As an example, an equilibrium in a 2-player game is a pair of strategies

(f_1, f_2) meeting the conditions

$$\begin{aligned} p_1(f_1, f_2) &\geq p_1(g_1, f_2) \\ p_2(f_1, f_2) &\geq p_2(f_1, g_2) \end{aligned}$$

for arbitrary strategies g_1 and g_2 . If the game is zero-sum, then $p_1(f_1, f_2) + p_2(f_1, f_2) = 0$.

2.2 The Game of Teams

This section introduces the *Game of Teams*, which was first defined by Menden-Deuer and Rowlett (2019) and later revised and refined in the publication by Menden-Deuer et al. (2021). It was originally designed to explain the vast diversity among asexually reproducing (cloning) microbes and microbial subspecies, which gives it a terminology reminiscent of biology. We call the ‘players’ individuals and we refer to the groups of players that constitute a team as ‘species.’ We may also use terms such as population size.

In a well-mixed population, every member has a constant, positive probability of meeting every other member. There are no closed groups but everyone has the same chance of meeting everyone. In a well-mixed population we may consider ‘teams’ or ‘species’ as labels and assign different characteristics to these teams. Imagine that two such teams constitute a population and that for each timestep the members of the population meet with a member from the other species and compare ‘strength,’ meaning that

- if one member is stronger than the other, the stronger one defeats the weaker and then replicates, or
- if both members are equally strong, it is a draw and both players remain in the game.

A draw results in no change to the size of each team, but if a member of one team defeats a member from the other team there is an increase by one to the first team while the other team is reduced by one member.

Imagine that these individual competitions take place simultaneously. If a member cannot be paired with another member, it waits until the rest of the members have compared strength. This is a timestep, or turn, in the game of teams.

Strength is here a positive, real number. An individual's strength is referred to as its *competitive ability*, in short CA. It should be assumed that the competing species, or teams, are on average not stronger than some positive number C . Otherwise it would be easy to win: just let all members of your team grow stronger. Thus, teams are characterized by the *distribution* of strength among the team members. We therefore define¹ strategies in the following way.

Definition 2.2.1. A *strategy* is a non-negative, Lebesgue-measurable, bounded function that is not identically zero, supported on a compact subset of $[x_0, \infty)$, for some real number x_0 .

We should think of strategies as assignment rules. At each turn, the strategy is the distribution of CA over the population in the sense that each player gets a CA randomly following the probability distribution given by the normalized strategy under the constraint that the average of the CAs is not greater than C . Sometimes we do not distinguish a team from its strategy but rather we use strategy and team interchangeably.

We will consider three classes of games. If no restrictions except those in Definition 2.2.1 are put on the strategies, we call this the *bounded game of teams* or the *bounded measurable game of teams*. If we consider continuous functions with support on a compact subset of $[x_0, \infty)$ we refer to this as the *continuous game of teams*. Let M be a real number and let a be an integer. Define for all non-negative integers j the fractions

$$x_j = \frac{j + a}{M}. \quad (2.2)$$

If we consider strategies from this set of x_j , we say that the game is the *discrete game of teams*. In this case we define strategies as follows.

¹Compare this with probability density functions; we assume not normalization but boundedness because 'strength' is never infinite.

Definition 2.2.2. A *discrete strategy* is a mapping n from $\{x_j\}_{j \geq 0}$ into the non-negative, real numbers such that $n(x_j)$ is not identically zero and only finitely many $n(x_j)$ are non-zero.

We will not distinguish discrete strategies if there is no risk of confusion, since it can be understood which definition that applies in most situations.

Example 2.2.3. Assume that two teams of three players each are playing cards with a special kind of deck: the cards can have five values 1, 2, 3, 4 or 5 and each team member can select one card as long as the average over the team is 3. Two or more players can select the same value. Each member pairs up with a member from the other team and the one having the highest value wins. What is the ‘best’ strategy in this situation? Let A be one team and B be the other. If A always plays with the number 3 while B selects 4 for two of its members and 1 for the remaining member, which team is expected to win? In each turn, the number of players in A that are expected to win are the number of players in B that play 1 or 2 and those players in B playing 4 or 5 will always win. Let $n_B(x_j)$ be the number of members of B that play j . Then

$$n_B(x_1) + n_B(x_2) - n_B(x_4) - n_B(x_5) \quad (2.3)$$

is the increase of members to team A at each turn. This sum equals $1 + 0 - 2 - 0 = -1$ meaning that A wins one player but also loses two players; B wins the first turn. The teams are not equally strong even though their average strength is the same. Team B wins because it has a smarter distribution of competitive ability among its members.

Let N_B denote the number of individuals in B . Interpreting $n_B(x_j)/N_B$ as the probability that a random player in B has the strength x_j , we can say that n_B is a strategy. It is in fact the number of players in B that are expected to play x_j . The team A plays with the strategy $n_A(x_3) = 3$ and $n_A(x_j) = 0$ for $j = 1, 2, 4, 5$.

The expectation in competition between teams

In general in the discrete game, the species A expects a population increase (or decrease) in competition with B given by

$$E[n_A, n_B] = \frac{\min\{N_A, N_B\}}{N_A N_B} \sum_{i \geq 0} n_A(x_i) \left(\sum_{i > j} n_B(x_j) - \sum_{i < j} n_B(x_j) \right) \quad (2.4)$$

where an empty sum is interpreted as zero. The factor in front of the sums accounts for the situation that one team is smaller than the other. If, say, $N_A = 100$ and $N_B = 99$ then

$$E[n_A, n_B] = \sum_{i \geq 0} \frac{n_A(x_i)}{N_A} \left(\sum_{i > j} n_B(x_j) - \sum_{i < j} n_B(x_j) \right)$$

so that $E[n_A, n_B]$ is in absolute numbers, ranging from -99 to 99 , which is exactly the minimal and maximal number of ‘lose’ and ‘win’ as one member of A cannot be paired with a member of B . If we are interested in the relative increase, we compute

$$p[n_A, n_B] = \frac{E[n_A, n_B]}{\min\{N_A, N_B\}}.$$

Game-theoretically, $p[n_A, n_B]$ is the payoff to team A in competition with B . The relative expectation, p , ranges from -1 to 1 . Passing between E and p simply means that we normalize the strategies.

In the continuous game and the bounded game, we define the expected win for a strategy f in competition with another strategy g as

$$E[f, g] = \alpha \int_{x_0}^{\infty} f(x) \left(\int_{x_0}^x g(y) dy - \int_x^{\infty} g(y) dy \right) dx \quad (2.5)$$

where

$$\alpha = \frac{\min \left\{ \int_{x_0}^{\infty} f(x) dx, \int_{x_0}^{\infty} g(x) dx \right\}}{\left(\int_{x_0}^{\infty} f(x) dx \right) \left(\int_{x_0}^{\infty} g(x) dx \right)}. \quad (2.6)$$

Again we define p as $p[f, g] = E[f/F, g/G]$ where F and G are the integrals of f and g over $[x_0, \infty)$, that is, we normalize them with respect to the

L^1 norm. Integration here means integration with respect to Lebesgue measure.

It is easily checked that both E and p satisfy the ‘zero sum dynamic’

$$p[f, g] + p[g, f] = 0 \quad (2.7)$$

in the bounded, the continuous and the discrete game. If more than two teams simultaneously compete we would like to consider one team in competition with the rest. Since

$$p[f, g_1 + g_2] = p[f, g_1] + p[f, g_2], \quad (2.8)$$

we may define the expected payoff to f in competition with a collection of strategies $\{g_i\}_{i=1}^n$ as $p[f, g_1 + \dots + g_n]$. The linearity property (2.8) does not hold for E due to the factor α , but nevertheless we define the expectation of f in competition with a collection of species $\{g_i\}_{i=1}^n$ as $E[f, g_1 + \dots + g_n]$ because any species competes against all the others (otherwise their members would belong to the same species).

An *equilibrium* (or *equilibrium point*) in the game of teams is defined as a collection of strategies $\{f_i\}_{i=1}^n$ that satisfy the following condition: For each $k \in \{1, 2, \dots, n\}$, if team no. k changes its strategy but all other teams retain their strategies then the payoff to team no. k does not increase. This means that for all $k \in \{1, 2, \dots, n\}$,

$$p[f_k, f_1 + f_2 + \dots + f_{k-1} + f_{k+1} + \dots + f_n] \geq p[g, f_1 + f_2 + \dots + f_{k-1} + f_{k+1} + \dots + f_n]$$

for any strategy g of the same type (bounded, continuous or discrete). The strategies that comprise an equilibrium point are known as *equilibrium strategies*.

This agrees with the definition of a Nash equilibrium. By the following proposition, it is equivalent to non-negative expectation in competition with any other strategy.

Proposition 2.2.1. *Assume that a collection of strategies $\{f_i\}_{i=1}^n$ is an equilibrium point. Then they satisfy*

$$p[f_j, f_k] = 0 \text{ for all } k, j, \quad \text{and} \quad p[f_k, g] \geq 0 \text{ for any strategy } g. \quad (2.9)$$

Equivalently, each f_k is an equilibrium strategy for the two-player game. Conversely, if $\{f_i\}_{i=1}^n$ satisfy (2.9) then $\{f_i\}_{i=1}^n$ is an equilibrium of the game.

A proof of this proposition is found in paper 1.

Using this proposition in the identification of equilibrium strategies we find those strategies f that satisfy $E[f, g] \geq 0$ for all strategies g . It is never important to retain the factor α and neither is it important to normalize the strategies, since any equation on the form $E[f, g] \geq 0$ is equivalent to $p[f, g] \geq 0$.

In paper 1, we use the notation \wp (this symbol is called the ‘Weierstrass p’) for the expectation E with $\alpha = 1$. We also use the notation $\wp[f; g_1, \dots, g_n] = \wp[f, g_1 + \dots + g_n]$. Again, using E or \wp does not change which strategies are equilibrium strategies, so we may use \wp instead of p .

A fair constraint on strategies. Translation invariance.

A team could always get better if all members would be stronger, that is, if the CAs of the members were allowed to grow without any constraint it would be trivial to construct a winning team. Assume therefore that only teams with an average CA of less than or equal to some positive number C are allowed to compete. We will refer to this as the constraint on the *mean competitive ability*, or MCA, and we define

$$\text{MCA}(n) = \frac{\sum_{k \geq 0} x_k n(x_k)}{\sum_{k \geq 0} n(x_k)} \quad (2.10)$$

in the discrete game, and in the bounded game as well as in the continuous game we define

$$\text{MCA}(f) = \frac{1}{\|f\|_{L^1}} \int_{\mathbb{R}} x f(x) dx, \quad \|f\|_{L^1} = \int_{\mathbb{R}} f(x) dx. \quad (2.11)$$

Notice that f is non-negative and $0 < \|f\|_{L^1} < \infty$ by the definition of a strategy. Integrating over \mathbb{R} is equivalent to integrating from x_0 to ∞ since strategies have compact support in $[x_0, \infty)$.

The game of teams is *translation invariant* and *scale invariant*. This means that given a collection of strategies we may assume that they are defined

on $[0, 1]$ or some other interval, say $[-1, 1]$. If f and g are two strategies, their support is contained in a closed interval $[a, b]$ such that $a < C < b$. Defining $\ell = b - a$, $\tilde{C} = (C - a)/\ell$ and

$$\tilde{f}(t) = f(\ell t + a), \quad \tilde{g}(t) = g(\ell t + a) \quad \text{for } t \in [0, 1] \quad (2.12)$$

we have

$$\begin{aligned} \text{MCA}(f) \leq C &\iff \text{MCA}(\tilde{f}) \leq \tilde{C}, \\ \text{MCA}(g) \leq C &\iff \text{MCA}(\tilde{g}) \leq \tilde{C} \end{aligned}$$

and $p[\tilde{f}, \tilde{g}] = p[f, g]$. This implies that the values of E and p are unchanged under the translations

$$x \mapsto x + x_0, \quad C \mapsto C + x_0, \quad (2.13)$$

with $x_0 \in \mathbb{R}$ in the bounded or continuous game, and correspondingly for the discrete game $j \mapsto j + a$, $C \mapsto C + x_a$ with $a \in \mathbb{Z}$. In other words, it is only the relative distance to C that matters for the outcome of the game. Therefore, it makes no difference to assume that the strategies are defined on a compact subset of $[0, \infty)$ or $[x_0, \infty)$ for some x_0 . For simplicity, one may assume that $x_0 = 0$.

One may of course ask whether it is necessary to have a lower bound at all on the CAs. The answer is yes, if we want the game to be interesting. If there is no lower bound on the CAs then we may always construct a species similar to the winning species in Example 2.2.3, that is one which sends one player to very low CAs and assign to the rest of its members a CA which is just slightly higher than those of the competing species'. A winning team is trivial to construct if the CAs are not bounded from below.

2.3 Main results of Paper 1

In paper 1 we identified the equilibria for the game of teams. Recall the definition of the fractions x_j in (2.2). In the discrete game, assume that the

MCA-constraint is such that C lies on some CA or precisely between CAs, that is,

$$C = \frac{2a + k}{2M} \quad \text{for some integer } k > 0. \quad (2.14)$$

Main results of paper 1

In the bounded game of teams, a strategy is an equilibrium strategy if and only if it is almost everywhere equal to, for some constant $K > 0$,

$$f(x) = \begin{cases} K, & x \in [x_0, 2C - x_0] \\ 0, & \text{otherwise.} \end{cases} \quad (2.15)$$

This is a constant, positive function with support on an interval centered around C . There are no equilibrium strategies in the continuous game of teams, because it would need to equal (2.15) which is not continuous. In the discrete case, A is an equilibrium strategy if and only if it is given by a constant $b > 0$ and if k is odd,

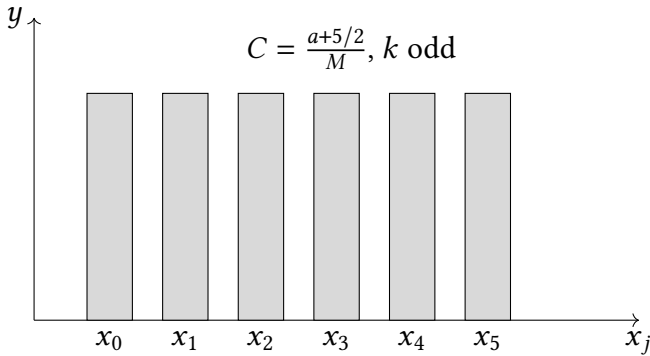
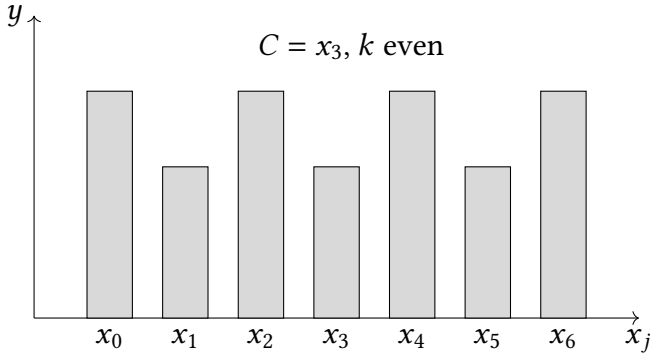
$$n_A(x_j) = \begin{cases} b, & 0 \leq j \leq k \\ 0, & \text{otherwise} \end{cases} \quad (2.16)$$

or if k is even, for b and c non-negative constants that are not both zero,

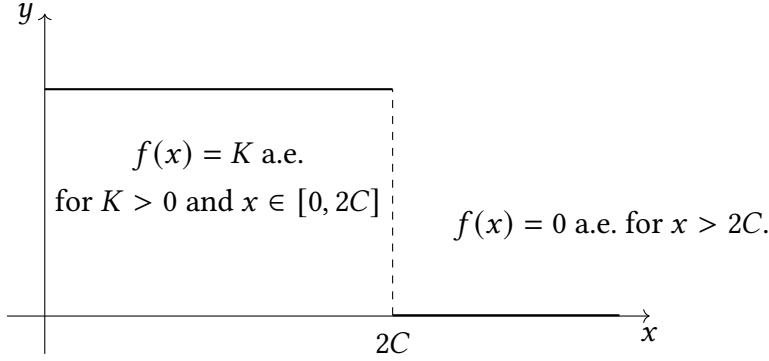
$$n_A(x_j) = \begin{cases} b, & 0 \leq j \leq k, j \text{ even} \\ c, & 1 \leq j \leq k - 1, j \text{ odd} \\ 0, & \text{otherwise} \end{cases} \quad (2.17)$$

In (2.17), not all n_A 's are zero by definition of a strategy but either b or c can be zero. Conversely, when there exists an equilibrium in the game of teams, it is comprised of the above strategies.

A vizualisation of equilibrium strategies are provided in the figures below.



The following is a visualization of the equilibrium strategies of the bounded game with $x_0 = 0$, which is no limitation due to translation invariance. Notice that this is a discontinuous function. In fact, this is the reason that there are no equilibrium strategies in the continuous game of teams. Any continuous function will lose against a continuous function that better mimics the equilibrium of the bounded game, but then no function will ever be protected against losing the game, so there is no equilibrium.



Proving these results requires computing the expectation between pairs of strategies and using Proposition 2.2.1. In the bounded game we show that $E[f, g] \geq 0$ for f given by (2.15) and any other strategy g by direct computation of E . As mentioned above, given any collection of strategies one can assume that they are defined on the interval $[0, 1]$ with $C \leq 1/2$ by the change of variables in (2.12). Then the continuous strategies and the discrete strategies satisfy the conditions of Theorem 1 in (Menden-Deuer et al., 2021). These are some of the main arguments that prove the results of paper 1.

Equilibrium strategies are positive and constant on an interval centered around C . In paper 1 we note that such functions distribute the CAs without favoring any values in particular. Every CA is equally probable, so the teams will have ‘maximal distribution’ of team members and all members are treated equally. This motivates the title of the paper, *The Strength of Diversity*.

Heuristically the main result of paper 1 may be summarized as follows:

A team can be guaranteed to not lose by assigning CAs to its members in such a way that all CAs within an interval centered on C are equally probable. In other words, a winning strategy is to create a maximally diverse team.

Chapter 3

Decision-making and disease outbreaks

A classical Game Theory case: People are not taking vaccines in the hope that everyone else would be vaccinated and they would be safe.

—Vineet Raj Kapoor (2021 on Instagram)

How many people will be infected in a population within, say, two weeks, given the current conditions such as the number of infected individuals and transmission rates? These types of questions are effectively answered by dividing the population into *compartments* (or categories). Changes to the categories are captured by dynamical models which are typically differential equations. This chapter introduces some of the standard models and presents our contribution to the field.

The COVID-19 pandemic outbreak in the beginning of 2020 issued many questions about people's behavior during diseases, such as 'What difference does it make if *this large* a portion of the population complies with recommendations while the rest do not?' My supervisor and I incorporated ideas from game theory with the disease spreading models SIR and SIS in order to provide some insights to these questions. We realized that the individual decisions during a pandemic resemble the prisoner's dilemma

(PD) and we investigated what happens when the population is under influence of the PD game while also being aware of the risk of catching the disease. This led to insights whether cooperation emerges when the game's payoff is a trade-off between the PD and the effect on disease spreading through changes to the infection transmission rate.

3.1 The early models: SIR and SIS

Some diseases confer immunity after an infection, while some do not. Immunity can last very long, sometimes as long as a lifetime, and sometimes much shorter. Examples of the former is the measles (Vårdguiden 1177.se, 2019). Common colds and many STIs are examples of diseases that do not confer immunity; one can catch the disease again almost immediately after recovery. (Niespodziana et al., 2012; Workowski et al., 2021)

The SIR model

The following model was an early attempt at describing the dynamics among the population during an outbreak of a disease that confers lifetime immunity or immediate death. The model cannot differentiate between dead and immune because both types mean that the individual – dead or immune – does not participate in the disease spreading.

Assume that the population is divided into three categories, each consisting of individuals that are in one of the following states:

- Susceptible
- Infectious
- Removed (dead or immune)

Denote by S , I and R the number of susceptible, infectious and removed individuals, respectively. Let N denote the size of the population, so that $S + I + R = N$. Dynamics in this model are captured by the change of state within the population, that is if, say, a susceptible individual gets sick then it moves to compartment I and correspondingly there would be an

increase by 1 of I while S decreases by 1.

The classic SIR model assumes that all individuals in the population has a certain, constant probability to get in contact with any of the other individuals. It further assumes that the probability of disease transfer from a infectious individual to a susceptible individual is a positive constant β . The number of susceptible individuals can only decrease since the infected individuals eventually become removed (by death or – hopefully – by becoming immune.) At each time, the expected decrease is proportional to the risk of a susceptible individual meeting with an infectious individual, that is, S would decrease by βSI . Let $S(t_k)$ be the number of susceptible individuals at time t_k . Then $S(t_{k+1}) - S(t_k) = \beta S(t_k)I(t_k)$ would describe this situation.¹



In large populations, it is usually feasible to approximate the integer numbers S , I and R with continuous real values. Therefore we describe the dynamics of S by means of the ordinary differential equation $\dot{S} = -\beta SI$, where the dot over S means differentiation with respect to the time variable. Then the flow scheme above says the the amount subtracted from S should be added to the compartment I . The rate at which individuals transfer to the removed state is such that $\dot{R} = \gamma I$ for a constant $\gamma > 0$. We obtain

$$\dot{S} = -\beta SI \quad (3.1)$$

$$\dot{I} = \beta SI - \gamma I \quad (3.2)$$

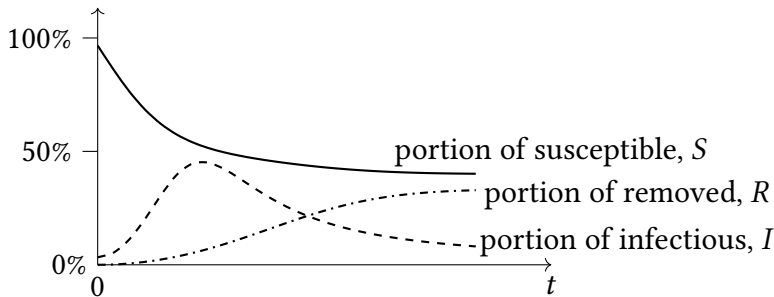
$$\dot{R} = \gamma I \quad (3.3)$$

Notice that $\dot{S} + \dot{I} + \dot{R} = 0$ meaning that the size of the population is constant.

A sketch of typical time dynamics in the SIR model is pictured below. The portion of susceptible (full line) decreases as the portion of infectious

¹Notice that β is dimensionless, although, since each timestep is indexed, we may argue that it has dimensions 1/(time).

(dashed line) increases during the outbreak of the disease. After some time, there is a portion of removed individuals (dashed dotted line). The dynamics stabilize after long time to a *steady state* with no infectious individuals. Notice that there might be susceptible individuals at steady state, because with $I = 0$ there is no flow between the compartments.



If we rewrite (3.2) as $\dot{I} = (R_0 S - 1)\gamma I$ with $R_0 = \beta/\gamma$ we see that the ratio R_0 determines whether there is a disease outbreak at small times. Above it is assumed that $R_0 S(0) > 1$ because otherwise there would be no increase of I at small times.

The SIS model

Instead of being ‘removed’ as in the SIR model we may assume that recovery from the disease is immediately followed by the risk of catching the same disease again. In other words, ending the disease is a *recovery* for the individual. This situation is correctly modeled if the recovered individuals ‘flow back’ to the susceptible compartment. That is, the portion γI is added at each time to the change of S , giving $\dot{S} = -\beta SI + \gamma I$. In total,

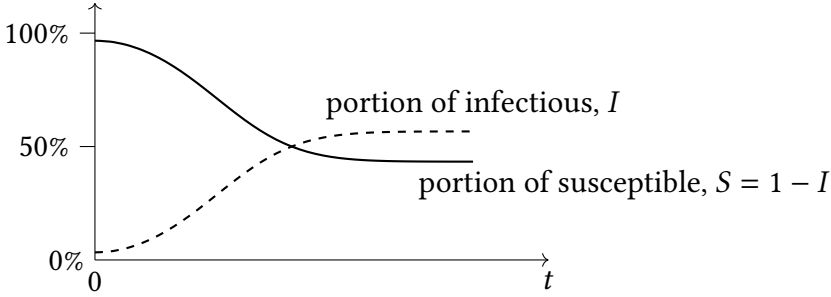
$$\begin{aligned}\dot{S} &= -\beta SI + \gamma I \\ \dot{I} &= \beta SI - \gamma I\end{aligned}$$

Since $S + I = 1$ in this situation, we can eliminate S from the equations. The full dynamics is described by

$$\dot{I} = \beta(1 - I)I - \gamma I \quad (3.4)$$

A sketch of typical time dynamics in the SIS model is pictured below. The portion of susceptible (full line) decreases as the number of infectious

(dashed line) increases during the outbreak of the disease. After some time, the dynamics stabilize to a *steady state* which is characterized by a balance between recovery and infection, that is, $\beta SI = \gamma I$.



Using (3.4) we compute

$$\frac{d}{dt} \frac{1}{I} = -\frac{\dot{I}}{I^2} \implies \frac{d}{dt} \frac{1}{I} + (\beta - \gamma) \frac{1}{I} = \beta. \quad (3.5)$$

If $\beta \neq \gamma$ then

$$I(t) = \frac{\beta - \gamma}{v_0 e^{-(\beta - \gamma)t} + \beta} \quad (3.6)$$

for some $v_0 \in \mathbb{R}$, cf. (Hethcote, 1989). If $\beta = \gamma$ then $I(t) = 1/(\beta t + w_0)$ for some $w_0 \in \mathbb{R}$. If $\beta > \gamma$ then $I(t) \rightarrow 1 - \gamma/\beta$ as $t \rightarrow \infty$ and if $\beta \leq \gamma$ then $I(t) \rightarrow 0$ as $t \rightarrow \infty$. The ratio between β and γ plays an important role in the dynamics, similar to the case of the SIR model.

More complex models

There is vast literature on more advanced models than the ones outlined above. For instance, one may assume that there is an incubation period for the disease. Then the susceptible individuals will not contribute to the disease spreading immediately after becoming infected, so there is a time delay between catching the disease and moving to the compartment I . Further examples of models are found in e.g. Vynnycky and White (2010).

3.2 Evolutionary game theory

Recall that a mixed strategy is a probability distribution over the options in a game. Playing with mixed strategies, it is standard notation to call

p_x the payoff for playing C with probability x . Assuming that the other player in a 2×2 matrix game uses the same strategy we can compute the expected payoffs. We will do this in the following section.

Replicator equations of symmetric games

In most classical applications, game theory focuses on decisions made by rational players using cognitive choice. On the contrary, the evolutionary application of games specifies a process of natural selection—individuals are merely the performers of an inherited program. In a 2×2 symmetric matrix game, a mixed strategy x may be subjected to a ‘replicator’s equation,’

$$\dot{x} = F(x), \quad (3.7)$$

which specifies the change of the strategy. A common model is assuming that the strategy changes in proportion to the payoff’s linear deviation from the mean, that is $F(x) = x(p_x - \langle p \rangle)$, where p_x is the payoff to x and $\langle p \rangle$ is the mean value of the payoffs.

Recall the payoff matrix from §1.1.1. We abstract away from the numbers of said matrix and denote the payoffs of the 2×2 symmetric game by R, T, P and S.

		player B	
		C	D
player A	C	R R	T
	D	S	P

The symmetric nature of the game implies that the game payoffs can be represented by the matrix

$$\begin{bmatrix} R & S \\ T & P \end{bmatrix}. \quad (3.8)$$

The entries are the payoffs to player A, where the top row is the payoff if A plays ‘cooperate.’ Now we can define what the prisoner’s dilemma (PD)

is mathematically: It is the situation

$$T > R > P > S. \quad (3.9)$$

Consider a replicator's equation with $F(x) = x(p_x - \langle p \rangle)$. In the symmetric matrix-form game we have

$$\begin{aligned} \dot{x} &= x(Rx + S(1-x) - Rx^2 - Sx(1-x) - Tx(1-x) - P(1-x)^2) \\ &= x(Rx(1-x) + S(1-x)^2 - Tx(1-x) - P(1-x)^2) \\ &= x(1-x)(Rx + S(1-x) - Tx - P(1-x)) \\ &= -x(1-x)((T-R)x + (P-S)(1-x)). \end{aligned} \quad (3.10)$$

We can write $\dot{x} = -x(1-x)(D_g x + D_r(1-x))$, where $D_g = T - R$ and $D_r = P - S$ (cf. Tanimoto, 2015, ch. 2). A special case of the PD game is the *Donor-Recipient game* which has $D_g = D_r$, meaning that the payoff advantage of defection over cooperation is independent of the opponent's choice. Denoting the payoff disadvantage by C (for 'cost') we write $D_g = D_r = C$. In the donor recipient game,

$$\dot{x} = -x(1-x)C. \quad (3.11)$$

Without loss of generality, one can assume that $P = 0$. In accordance with the PD-condition (3.9) we require

$$-C < 0 < R < R + C \quad (3.12)$$

and the payoff matrix becomes

$$\begin{bmatrix} R & -C \\ R + C & 0 \end{bmatrix}. \quad (3.13)$$

In the next sections, we will formulate the dynamics using R and C only, and forget S , T and P .

Network reciprocity

Networks in games represent contacts between players. Only neighbors can interact, that is, in the context of disease spreading there are a limited number of possible transmission paths of the disease. We make the following assumptions:

1. What two-steps neighbors do is irrelevant.
2. We assume the graph is degree-regular, meaning that each node has the same number, k , of connecting edges.

Under these assumptions the cost C in the donor recipient game decreases by $N(k)$, where

$$N(k) = \frac{Rk - 2C}{(k+1)(k-2)}, \quad k \neq 2, \quad N(2) = R$$

(Tanimoto, 2015; Ohtsuki and Nowak, 2006). Notice that $N(k) \rightarrow 0$ if $k \rightarrow \infty$, which reflects the fact that unlimited network connectivity recovers the network-free, well-mixed situation. Now the equation (3.10) modifies by the change of the payoff matrix to

$$\dot{x} = -x(1-x)(C - N(k)). \quad (3.14)$$

By (3.12), the game is of PD type when $N(k) - C < 0 < R$, which is equivalent to

$$0 < R < C(k-1). \quad (3.15)$$

That is, the benefit of mutual cooperation compared to the cost of mitigation determines whether the game is of PD type. For example, when R/C is large, corresponding to low costs of mitigation and/or high benefit of mutual cooperation, the game may cease to be of PD type for values of sufficiently small values of k such that (3.15) breaks. For sufficiently large values of k – in particular for well-mixed populations – the game is always of PD type.

3.2.1 The dynamical beta-parameter

In the previous section we encountered the equations of the SIR model and the SIS model, which both assume that there is a infection rate β such that βSI is the portion of susceptible individuals that catch the disease at each timestep. The parameter β need not be constant. In our work, we have assumed that some individuals in the population are more effective at disease spreading, meaning that we associate a larger infection rate to

them. We denote those individuals by D and the rest get the label C , so that β_D is the infection rate of individuals of class D and β_C is the infection rate of individuals of class C . By assumption $\beta_C < \beta_D$. Let β be the average over the population, that is if x is the portion of C -individuals then

$$\beta = x\beta_C + (1 - x)\beta_D. \quad (3.16)$$

Assuming that the infection rate is determined by individual *choice*, we may interpret D as the game theoretic strategy ‘defection.’ Similarly, C stands for ‘cooperation.’

In order to model trends in the choice between cooperation and defection within the population, we assumed that the portion x is governed by a replicator’s equation. The payoffs that each individual perceive is assumed to be a balance between contributing to the common good (by cooperating) or gaining personal benefits (by defecting). In this situation x depends on time and we write $x = x(t)$.

By interpreting x as the portion of cooperators there is a flow of the defectors becoming cooperators given by

$$\dot{x} = \alpha x(1 - x)(\beta_D - \beta_C)I, \quad (3.17)$$

where $\alpha \geq 0$ is a parameter. The factor $x(1 - x)$ ensures that x remains in the range $0 \leq x \leq 1$. We may interpret (3.17) as a risk assessment managed by each individual: If there are many more infectious individuals that do not cooperate, then there is more risk of catching the disease and more individuals will decide to cooperate, that is, x will increase.

3.2.2 The combined SIR-PD and SIS-PD models

We would like to consider the total contribution of (3.17) and (3.14). Notice that both equations are on the same form as (3.11). Individuals are assumed to consider the total cost as the sum of $-\alpha(\beta_D - \beta_C)I$ and $C - N(k)$. The portion x is thus determined by the choices of individuals making conscious decisions based on the risk of catching the disease on the one hand, and on the other hand the benefits of the PD-game. These decisions determine

the total transmission rate of the disease, which the entire population experiences. In the SIR-PD model,

$$\dot{S} = -((1-x)\beta_D + x\beta_C)SI \quad (3.18)$$

$$\dot{I} = ((1-x)\beta_D + x\beta_C)SI - \gamma I \quad (3.19)$$

$$\dot{R} = \gamma I \quad (3.20)$$

$$\dot{x} = x(1-x)(\alpha(\beta_D - \beta_C)I - (C - N(k))) \quad (3.21)$$

whereas in the SIS-PD model,

$$\dot{I} = ((1-x)\beta_D + x\beta_C)(1-I)I - \gamma I \quad (3.22)$$

$$\dot{x} = x(1-x)(\alpha(\beta_D - \beta_C)I - (C - N(k))) \quad (3.23)$$

Notice that the portion of cooperators is influenced by I which is itself dynamic. This results in a feedback mechanism. If a member of the population decides to change strategy from defect to cooperate, then the effective disease transmission rate (3.16) is decreased which causes I to shrink which in turn causes x to increase. Nevertheless, there are steady state solutions to the SIR-PD model and the SIS-PD model as we will see in the next section.

The parameter α balances the contribution from the term $(\beta_D - \beta_C)I$ with that of the term $C - N(k)$. We interpret this as a timescale difference. If an individual gets updates on the portion of infectious, I , on the timescale ‘days’ whereas the PD game payoff can be received on the timescale ‘hours’ then the PD payoffs have more influence on this player’s decision. In that case, α is small. Conversely, if α is large then the information about I is acquired frequently and the player acts accordingly. In this interpretation, the non-negative sign on α can be thought of as correctness in the player’s risk assessment. If the player thinks that a disease is dangerous and if the source of information is reliable, then the rational and conscious player acts with $\alpha \geq 0$.

3.3 Main results of Paper 2

Heuristically the main result of paper 2 may be summarized as follows:

For a communicable disease that do not infer immunity, the quicker individuals access information about the portion of infectious individuals in the population the lower is the portion of infectious individuals at steady state. However, in the context of diseases that infer immunity, no one cooperates at steady state.

SIS-PD equilibrium points

Recall that the SIS-PD model is defined by the set of equations (3.22)–(3.23) with $\beta_C < \beta_D$.

Theorem 3.3.1. *The equilibrium points of the SIS-PD system are*

$$(x, I) \in \left\{ (0, 0), (1, 0), \left(0, 1 - \frac{\gamma}{\beta_D}\right), \left(1, 1 - \frac{\gamma}{\beta_C}\right), (x^*, I^*) \right\}, \quad (3.24)$$

where

$$x^* = \frac{\beta_D}{\beta_D - \beta_C} - \frac{\gamma}{(\beta_D - \beta_C)(1 - I^*)}, \quad I^* = \frac{(C - N(k))}{\alpha(\beta_D - \beta_C)} \quad (3.25)$$

The equilibrium points are well-defined and stable under the conditions given in the following table.

Equilibrium	Condition
(0, 0)	$\beta_D < \gamma, \alpha(C - N(k)) > 0$
(1, 0)	$\beta_C < \gamma, \alpha(C - N(k)) < 0$
$(0, 1 - \gamma/\beta_D)$	$\beta_D > \gamma, \alpha \leq \check{\alpha}$
$(1, 1 - \gamma/\beta_C)$	$\beta_C > \gamma, \alpha \geq \hat{\alpha}$
(x^*, I^*)	$\hat{\alpha} < \alpha < \check{\alpha}, 0 < I^* < 1, 0 < x^* < 1$

Here,

$$\check{\alpha} = \frac{\beta_D}{\beta_D - \gamma} \frac{C - N(k)}{\beta_D - \beta_C} \quad \text{and} \quad \hat{\alpha} = \frac{\beta_C}{\beta_C - \gamma} \frac{C - N(k)}{\beta_D - \beta_C}. \quad (3.26)$$

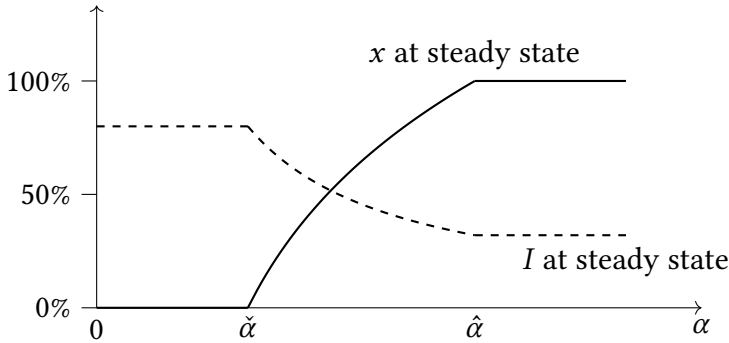
Since I denotes a portion we must have $0 \leq I \leq 1$ which is one main criterion for the steady states to be well-defined. The same holds for x .

Proving the stability of each steady state solution also involves computing the Jacobian matrix of the SIS-PD equations. They are stable if the eigenvalues of the Jacobian matrix are negative (Chione, 2009).

Following the values of α , the equilibrium *transitions* between the asymptotically stable equilibrium points of Theorem 3.3.1. The following table shows the equilibria in three regions of α assuming $\gamma < \beta_C < \beta_D$. The transitions occur at $\check{\alpha}$ and $\hat{\alpha}$, which are defined in (3.26).

Range	Equilibrium
$\alpha \leq \check{\alpha}$	$(0, 1 - \gamma/\beta_D)$
$\check{\alpha} < \alpha < \hat{\alpha}$	(x^*, I^*)
$\hat{\alpha} \leq \alpha$	$(1, 1 - \gamma/\beta_C)$

The following figure is a visualization of the values in the above table.



As α increases, the population moves from defection ($x = 0$) to partial cooperation to cooperation ($x = 1$). At the same time, I , the portion of infectious individuals is decreasing. If α is sufficiently large and the transmission rate for cooperating individuals, β_C , decreases (still keeping $\gamma < \beta_C$) then the portion of infectious individuals tends to zero.

SIR-PD equilibrium points

Recall that the SIS-PD model is defined by the set of equations (3.18)–(3.21). Computing the Jacobian of this system, we identify the stable steady states precisely like in the above SIS-PD situation. Since $S + I + R = 1$, the quantity R at equilibrium is given by $R^* = 1 - S^*$ in the following theorem.

Theorem 3.3.2. *The SIR-PD model always stabilizes to a set of equilibrium points (x^*, I^*, S^*) with*

$$x^* \in \{0, 1\}, \quad I^* = 0, \quad 0 \leq S^* \leq 1. \quad (3.27)$$

All of these are possible; the exact values are determined by initial conditions. The equilibrium points with $x = 0$ are stable if $\beta_D S \leq \gamma$. If $\beta_D S < \gamma$ the equilibrium point is unstable. All equilibrium points with $x = 1$ are unstable.

Only the outcome with no cooperation whatsoever, $x^* = 0$, is stable in the SIR-PD model, which is actually expected because the SIR model ‘empties’ the I -compartment. At steady state there are no infectious individuals left; the SIR model terminates at a no-disease state. Therefore, there is nothing to gain from cooperating. Looking at the dynamics before steady state, however, there is an increase of cooperators during the outbreak of the disease.

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